Modelling Traffic Flow: Solving and Interpreting Differential Equations

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Abstract

A simple mathematical model for how traffic flows along a road is introduced. The resulting first-order ordinary differential equations can be used as an application of solution techniques taught at A-level and first year undergraduate level, and as a motivator to encourage students to think critically about the physical interpretation of the results which the equation produces.

Introduction

WHEN students study differential equations they are typically introduced to a range of important applications from physical science, such as simple harmonic motion (SHM), Newton's law of cooling and radioactive decay. The student is rarely encouraged to reflect critically on the nature of these mathematical models and this is probably for two main reasons. First, models such as these noted above give very good representations of the corresponding basic physical processes and so there isn't a lot for the student to be critical about. Secondly, the physical processes being modelled tend to be somewhat divorced from the students' everyday experience and so it can be difficult for the students to feel that they are in a position to comment.

A topic such as how traffic flows along a stretch of road is, however, well within the experience of all students. It is also a topic of substantial research interest to a whole range of academics such as mathematicians, civil engineers, geographers, ecologists and management scientists and has been modelled in a number of ways utilising ideas from areas such as fluid flow,[1] statistical physics[2] and chaos theory.[3] This range of approaches indicates the complexity of the problem. In this paper, however, we shall introduce a very basic model which results in a first-order ordinary differential equation that can be solved using the integrating factor or complementary function plus particular integral method. The model could be usefully utilised in teaching in a number of ways; as an application of differential equations, or as a motivator to the discussion of the ideas of mathematical modelling, or as an investigative project into the properties of the model.

Models for traffic flow

Consider a single one way stretch of road with a single entrance for traffic to flow onto the road and a single exit for traffic to flow off. Such a stretch of road is usually referred to as a *single link* (a two way stretch of road between two points A and B would be represented by two links; one going from A to B and the other from B to A).

As mentioned in the Introduction, there are many highly sophisticated mathematical models for the flow of traffic along a single link, and these models fall into the two broad categories of macroscopic and microscopic. In microscopic models the motion of each individual vehicle is simulated in relation to the motion of the driver in front (this class of models is referred to as car following models[1]). Macroscopic theories, however, ignore the behaviour of individual drivers and instead attempt to model gross structures of traffic flow in terms of properties such as traffic density (vehicles/metre) and flow (vehicles/second). In both cases the range of models which have been developed extends from the extremely simple[4] to the extremely complex.[5]

The mathematician's idea of perfect road users

One of the things which makes the succcessful mathematical modelling of traffic so difficult is that, at its root, what we are trying to model is human driver behaviour; and all humans are different. Drivers have different preferred "cruising" speeds, react differently to the behaviour of their nearest neighbour drivers who are directly in front and behind them, and some drivers even brake for no apparently good reason while on a clear road in perfectly good conditions! If only drivers would behave in a more uniform way the modelling job would be much simpler. So what is the mathematician's idea of the perfect set of road users? Well, quite simply, it would be a set of drivers who all drive at exactly the same speed along a link (irrespective of how close together they are) and so take exactly the same time (which we call the trip time, τ) to travel from entrance to exit. This would mean, in terms of a macroscopic model, that for an initially empty road with inflow $u(t)$ (vehicles/unit time) entering the link the outflow from the exit would be given by $q(t)$ (vehicles/unit time) where,

$$q(t) = u(t - \tau). \tag{1}$$

Thus the outflow is just the inflow shifted by the trip time τ. We will refer to equation (1) as the *perfect road user model.*

A differential equation model for traffic flow

A slightly more complex macroscopic model for traffic flow along a single link assumes that the rate of outflow q from the link is a function of the total number (or volume, v) of vehicles on the link at time t. By conservation of vehicles the rate of change of the volume of vehicles on the link must equal the inflow–the outflow or,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = u(t) - q(v). \tag{2}$$

In this case the model consists in solving (2) for physically reasonable forms of $q(v)$.

Choosing the form of $q(v)$; uncongested traffic flow

Any road will have a typical light traffic trip time, τ, associated with it. This trip time is the time it will take a typical vehicle to traverse the link from entrance to exit in uncongested conditions. Obviously, if the volume of traffic on the link becomes too large we would expect the trip time to increase as the link becomes congested. But initially we shall ignore this effect and assume that the link can allow traffic to flow freely irrespective of the number of vehicles on it at any given time. Of course if such a mythical road existed it would be the answer to every Minister of Transport's dreams. On such a road, traffic jams would never occur and the ever increasing volume of traffic on roads, which is the cause of the increased stress levels of many an urban commuter, would cause no problem. But, as we shall see, in the context of our differential equation model such a road is not quite as marvellous as it might appear.

Consider a link with a volume v vehicles distributed evenly along it. If the traffic moves freely through the link we would expect the last vehicle to exit the link after time τ and so the average outflow from the link would be given by (volume of traffic exiting)/(time taken) $= v/\tau$. From this we postulate the uncongested form of $q(v)$ as

$$q(v) = \frac{1}{\tau} v. \tag{3}$$

Substituting (3) into (2) gives

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{\tau} v = u(t). \tag{4}$$

We refer to equations (3) and (4) as the *uncongested flow model* and to illustrate its behaviour we consider a number of examples.

Linearly increasing inflow and the ultimate in "road rage"

Consider the general linear inflow,

$$u(t) = \alpha + \beta t \tag{5}$$

where α, $\beta \geq 0$ and $v(0) = 0$.

Substituting (5) into (4) and using (3) gives the outflow from the link as

$$\begin{aligned} q(t) &= \alpha + \beta(t - \tau) - (\alpha - \beta\tau)\,\mathrm{e}^{-(1/\tau)t} \\ &= u(t - \tau) - (\alpha - \beta\tau)\,\mathrm{e}^{-(1/\tau)t}. \end{aligned} \tag{6}$$

Notice that

$$\lim_{t\to\infty} [q(t)] = u(t - \tau). \tag{7}$$

Thus, in this case, the asymptotic form of the outflow is simply given by the inflow shifted by the trip time τ, i.e. the traffic behaves asymptotically as it it were made up of "perfect road users". Equation (6) is illustrated in Figure 1 for the case where $u(t) = 10t$ and the trip time $\tau = 10$.

Figure 1 also illustrates a physically bizarre property of the uncongested flow model, which holds irrespective of the form of the inflow and the link trip time τ; namely that the onset of an inflow generates (instantaneously) an outflow from the exit. Mathematically, this results from the fact that the outflow, q, is a function of the volume of traffic, v, on the link with no regard to where along the link the traffic actually is (i.e. the model does not take any account of the spatial variation along the length of the road). Physically it can be interpreted as the model instantaneously and uniformly spreading inflowing traffic along the entire length of the link and hence allowing the vehicles which enter at $t = 0$ to move from the entrance to exit at infinite speed. This makes for a particularly nasty form of "road rage" where drivers decide to take advantage of the clear road ahead and drive as fast as they can, and worse still, it turns out that they have unusually high performance cars!

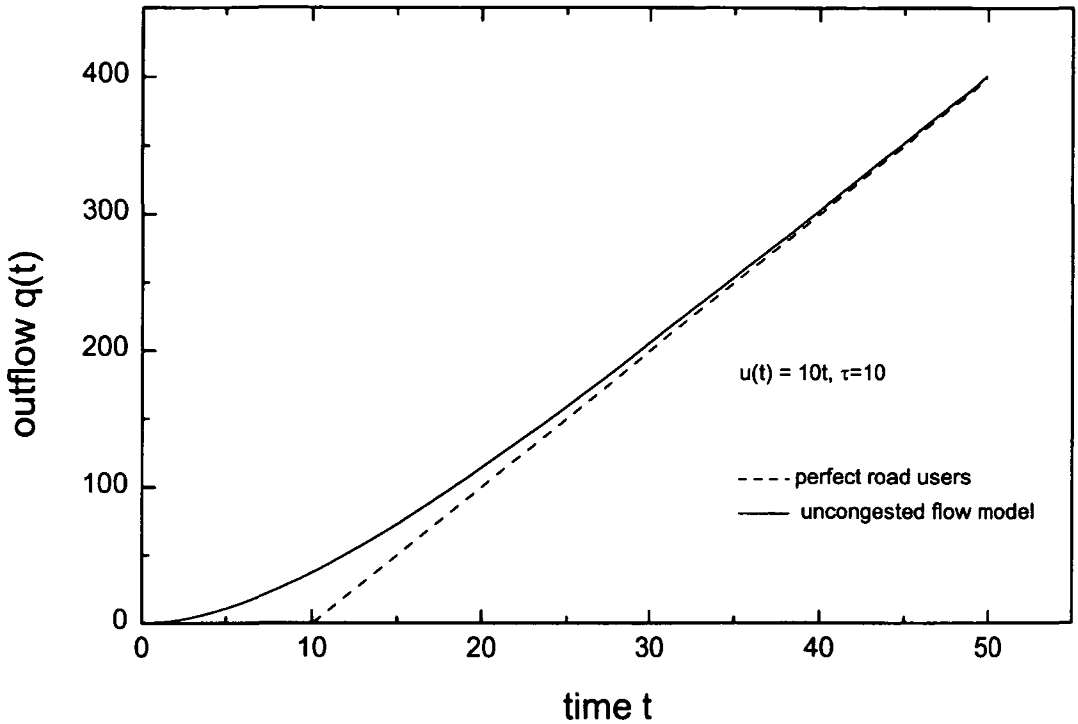

Fig. 1. Comparison between perfect road user model and uncongested flow model for linearly increasing traffic inflow

Journeys which take forever

Consider thee case where there is an initial volume of traffic. V_0, on the link and there is no inflowing traffic, i.e. we solve (4) subject to $u(t) = 0$ and $v(0) = V_0$, and use (3) to obtain

$$q(t) = \frac{V_0}{\tau} e^{-(1/\tau)t}. \qquad (8)$$

Note that the model predicts that the time required for all the traffic to clear the link is infinite! Physically we can interpret this as meaning that vehicles at the "tail end" of the block of traffic move infinitely slowly. Even with this being the case, however, it is easy to show that the average time, $\bar{T}$, spent by a vehicle on the link is the trip time τ, i.e.

$$\bar{T} = \frac{\int_0^\infty t v(t)\,dt}{\int_0^\infty v(t)\,dt} = \tau \qquad (9)$$

where, substituting equation (8) into equation (3),

$$v(t) = V_0 e^{-(1/\tau)t}. \qquad (10)$$

Thus we have two general predictions from our uncongested flow model so far. First (from the previous section), traffic entering an initially clear road at $t = 0$ moves along the road infinitely quickly and secondly, the final piece of traffic entering the road moves infinitely slowly! Before becoming too concerned with the existence of a road on which you may be trapped forever, or on which you may move superluminally depending on the somewhat demonic whim of where you are in the queue of traffic, let's look at another example.

Blocks of traffic

Consider a "block" of traffic of the form

$$u(t) = \begin{cases} 0 & t < 0 \\ u_0 & 0 \le t < T \\ 0 & t \ge T \end{cases} \qquad (11)$$

The uncongested flow model predicts outflowing traffic of the form,

$$q(t) = \begin{cases} 0 & t < 0 \\ u_0(1 - e^{-(t/\tau)}) & 0 \le t < T. \\ u_0(1 - e^{-(T/\tau)})\, e^{(T-t)/\tau} & T \ge t \end{cases} \qquad (12)$$

Figure 2 shows results generated by equation (12) for a range of trip times τ with $u_0 = 100$ and $T = 30$. Note that the outflows have been shifted along the time scale to allow direct comparison

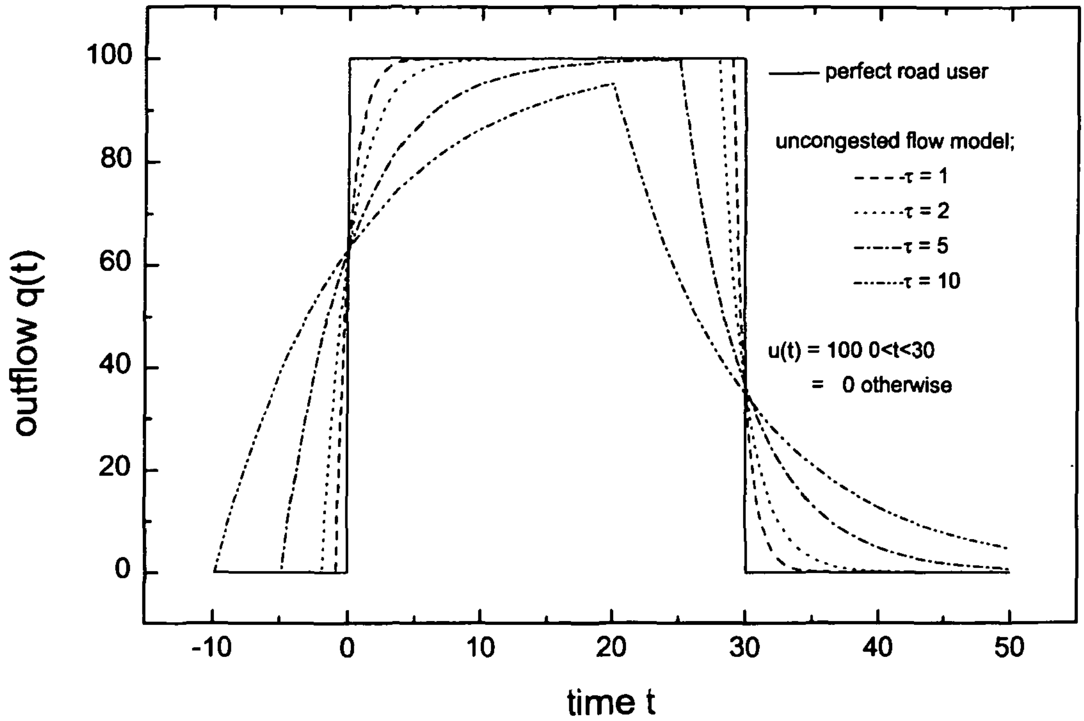

Fig. 2. Comparison between perfect road user model and uncongested flow model for a block of inflowing traffic.

with the perfect road user model. Two points are worth noting from Figure 2.

First, as noted earlier, the uncongested flow model predicts traffic at the start of the block of traffic moves infinitely quickly and traffic at the end of the block moves infinitely slowly. Secondly, as $\tau \to 0$ the outflows become more like the results of the perfect road user model. Or conversely, as τ increases we can regard the outflows from the uncongested flow model as resulting from an increased "diffusing" or "spreading" of the perfect road user model.

Such "diffusing" and "spreading" of traffic profiles does, of course, actually happen in real life, and so here we have he model predicting a real traffic phenomenon. This behaviour finds its root in the model assumption that outflow, q, is a function of link volume only. As mentioned before, physically this can be interpreted as the model instantaneously and uniformly spreading inflowing traffic along the length of the link. Thus although the uncongested flow model predicts (at least qualitatively) a real phenomenon it does so from use of a crude physical assumption and so we wouldn't expect good quantitative agreement at this point between the model and the real world.

Congested traffic: death by traffic jam

In the uncongested flow model the volume of traffic on the link can increase without bound and the link will never become congested, because increased volumes of traffic will never adversely affect traffic outflow. This is, of course, not what we would expect to happen in real life. As the link volume increases we would expect flow to become hindered and eventually jam. A simple way to model this is with the quadratic function

$$q(v) = \begin{cases} \dfrac{1}{J\tau} v(J - v) & 0 \le v \le J \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where J is the "jam volume" at which traffic flow stops. Note that for small values of v (or large values of J) the quadratic model behaves like the uncongested flow model. Also note that the outflow from the link cannot exceed the maximum value of $J/4\tau$. We refer to equations (2) and (13) as the *congested flow model.*

The differential equation which results from substituting (13) into (2) has an analytic solution only in the case of very simple inflows. As an example we consider the case of constant inflow where

$$u(t) = \begin{cases} 0 & t < 0 \\ U_0 & t \geq 0 \end{cases} \tag{14}$$

and the initial volume of traffic on the link is zero, i.e. $v(0) = 0$.

Substituting (14) and (13) into (2) and rearranging and integrating gives

$$\frac{1}{J\tau} t = \int_0^v \frac{\mathrm{d}x}{U_0 J\tau - Jx + x^2}. \tag{15}$$

The form of the solution of this integral depends on the sign of the discriminant of the quadratic in the integrand.[6] Solving (15) for $v(t)$ and then back substituting into (13) to find the outflow q gives the following solutions.

(a) For $\alpha^2 = J^2 - 4U_0 J\tau > 0$

$$q(t) = \frac{2U_0 J \tanh\left(\frac{\alpha t}{2J\tau}\right)}{\left(\alpha + J\tanh\left(\frac{\alpha t}{2J\tau}\right)\right)^2} \times \left(\alpha + (J - 2U_0\tau)\tanh\left(\frac{\alpha t}{2J\tau}\right)\right) \tag{16}$$

where

$$\alpha = \sqrt{J^2 - 4U_0 J\tau}. \tag{17}$$

Note that

$$\lim_{t\to\infty} [q(t)] = U_0. \tag{18}$$

Thus as $t \to \infty$ the steady state outflow is equal to the inflow.

(b) For $\alpha^2 = J^2 - 4U_0 J\tau = 0$

$$q(t) = U_0\left(1 - \left(\frac{2\tau}{2\tau + t}\right)^2\right). \tag{19}$$

And again, as $t \to \infty$ the steady state outflow equals the inflow.

(c) For $\alpha^2 = J^2 - 4U_0 J\tau < 0$

$$q(t) = \frac{1}{J\tau}\left(\left(\frac{J}{2}\right)^2 + \left(\frac{\alpha}{2}\right)^2 \times \tan^2\left[\frac{\sqrt{-\alpha^2}}{2}\left(\frac{t}{J\tau}\right) - \tan^{-1}\left(\frac{J}{\sqrt{-\alpha^2}}\right)\right]\right). \tag{20}$$

In this case the link will jam (i.e. $q(t) = 0$) at time

$$T_{\text{jam}} = \frac{4J\tau}{\sqrt{-\alpha^2}} \tan^{-1}\left(\frac{J}{\sqrt{-\alpha^2}}\right). \tag{21}$$

Thus, in the case of a constant inflow U_0, for a link with given jam volume J and light traffic trip time τ, the link will jam in time T_{jam} if

$$U_0 > J/4\tau. \tag{22}$$

Further, given the form of (13), if the link jams then it will *never* "unjam", even if the inflow is stopped. Once the link volume reaches J outflow stops, and there is no mechanism in the model to allow the link volume to fall below J again. Thus the link remains jammed for all future time with an ever increasing volume of inflowing traffic, producing the worst possible nightmare scenario for the motorist: an eternal traffic jam with the potential of an increasing multi-vehicle pile up.

This situation can be averted by stopping the inflow of traffic at a time $t < T_{\text{jam}}$, thus allowing the link to clear before a jam occurs.

Figure 3 shows the effect of reducing the jamming volume, J, on a link. In Figure 3, $U_0 = 100$ and $\tau = 10$ and so from equation (22) the critical jam volume, below which the link will experience a jam is $J = 4000$. Note also in Figure 3 the uncongested flow model is equivalent to the congested flow model with $J = \infty$, and so as J is increased the results from the congested model approach those of the uncongested model.

A numerical example

In this section we apply the models as set out above to some realistic data cast in the form of a question which could conveniently be used in class.

Question

Consider an initially empty, 5 km stretch of road. At time $t = 0$ traffic begins to flow onto the road at a constant rate of 1.2 vehicles/second and with speed of 72 km/hr (≡20 m/s). The average length of each vehicle is 5 m.

(a) Under such conditions does the *congested flow model* predict that the road will jam? If so, when?

What is the outflow predicted by

(b) the *perfect road user model*,

(c) the *uncongested flow model*,

(d) the *congested flow model*,

10 minutes after inflow commences?

Solution

First, we lay out the values of the parameters which we require, namely inflow rate, U_0, trip time, τ, and jam volume, J. We are given $U_0 = 1.2$ vehicles/second. We take $\tau = 250$ s, which

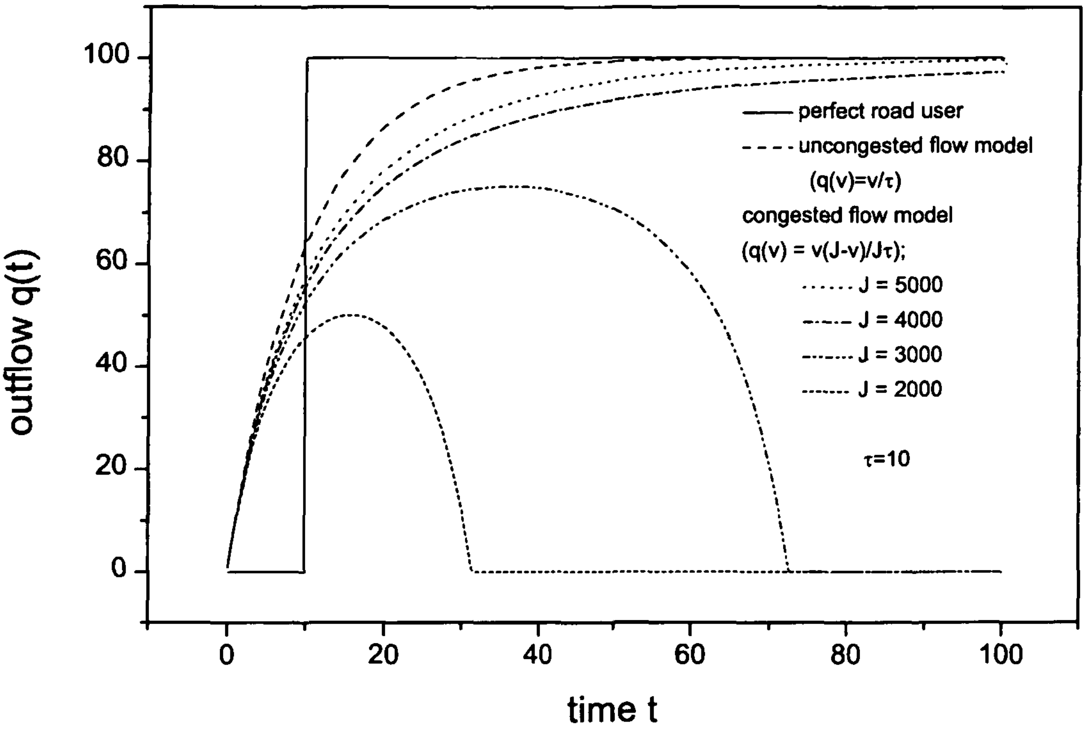

Fig. 3. Comparison between perfect road user model and uncongested and congested flow models for a constant inflow of traffic.

is simply the time taken for a vehicle travelling at 20 m/s to travel 5000 m from entrance to exit of the road. Finally we set $J = 1000$, which is the maximum number of vehicles, of length 5 m, which can be placed "bumper to bumper" along 5000 m of road.

(a) Using (22), the *congested flow model* jams if $U_0 > J/4\tau$, which is true for the above data. Thus jamming does occur, and using (21), the jam begins at $T_{\text{jam}} \approx 43$ minutes.

We can easily evaluate the predicted outflows from the 3 models at $t = 600$ seconds using (1), (12) and (20) to give,

(b) 1.2 vehicles/second, (c) 1.09 vehicles/second, and (d) 0.9 vehicles/second.

Concluding remarks

The simple traffic flow models that have been considered in this paper will provide students with an opportunity to exercise their skills in differential equations and integration. They can also be used to encourage students to look critically at the solutions they obtain and interpret them physically. Hopefully this latter process is aided by the fact that the models predict novel phenomena such as infinite journey time, "road rage" and the prospect of "death by traffic jam". Finally, the best way to improve these simple models is to include spatial dependence and thus move from the solution of an ordinary differential equation with variables of time, flow and volume to the solution of a partial differential equation with variables of space, time, flow and density. This more realistic formulation of the problem is commonly known as the Lighthill, Whitham[1] or Lighthill, Whitham and Richards model.[7]

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